Study Material for B.Sc.(H)(II Sem.)

Mathematics (Unit III)

3 THE METHOD OF UNDETERMINED COEFFICIENTS

A. Introduction; An Illustrative Example

We now consider the (nonhomogeneous) differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x), \tag{4.35}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are constants but where the nonhomogeneous term F is (in general) a nonconstant function of x. Recall that the general solution of (4.35) may be written

$$y = y_c + y_p,$$

where y_c is the *complementary function*, that is, the general solution of the corresponding homogeneous equation (Equation (4.35) with F replaced by 0), and y_p is a *particular integral*, that is, any solution of (4.35) containing no arbitrary constants. In Section 4.2 we learned how to find the complementary function; now we consider methods of determining a particular integral.

We consider first the method of *undetermined coefficients*. Mathematically speaking, the class of functions F to which this method applies is actually quite restricted; but this mathematically narrow class includes functions of frequent occurrence and considerable importance in various physical applications. And this method has one distinct advantage—when it *does apply*, it is relatively simple!

▶ Example 4.29: Introductory Example

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x} \tag{4.36}$$

We proceed to seek a particular solution y_p; but what type of function might be a possible candidate for such a particular solution? The differential equation (4.36) requires a solution which is such that its second derivative, minus twice its first derivative, minus three times the solution itself, add up to twice the exponential function e^{4x}. Since the derivatives of e^{4x} are constant multiples of e^{4x}, it seems reasonable that the desired particular solution might also be a constant multiple of e^{4x}. Thus we assume a particular solution of the form

$$y_p = Ae^{4x}, \tag{4.37}$$

where A is a constant (undetermined coefficient) to be determined such that (4.37) *is* a solution of (4.36). Differentiating (4.37), we obtain

$$y_p' = 4Ae^{4x} \quad \text{and} \quad y_p'' = 16Ae^{4x}.$$

Then substituting into (4.36), we obtain

$$16Ae^{4x} - 2(4Ae^{4x}) - 3Ae^{4x} = 2e^{4x}$$

or

$$5Ae^{4x} = 2e^{4x}. \tag{4.38}$$

Since the solution (4.37) is to satisfy the differential equation identically for *all* x on some real interval, the relation (4.38) must be an identity for all such x and hence the coefficients of e^{4x} on both sides of (4.38) must be respectively equal. Equating these coefficients, we obtain the equation

$$5A = 2,$$

from which we determine the previously undetermined coefficient

$$A = \tfrac{2}{5}.$$

Substituting this back into (4.37), we obtain the particular solution

$$y_p = \tfrac{2}{5}e^{4x}.$$

Now consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{3x} \tag{4.39}$$

which is exactly the same as Equation (4.36) except that e^{4x} in the right member has been replaced by e^{3x}. Reasoning as in the case of differential equation (4.36), we would now assume a particular solution of the form

$$y_p = Ae^{3x}. \tag{4.40}$$

Then differentiating (4.40), we obtain

$$y_p' = 3Ae^{3x} \quad \text{and} \quad y_p'' = 9Ae^{3x}.$$

Then substituting into (4.39), we obtain

$$9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 2e^{3x}$$

or

$$0 \cdot Ae^{3x} = 2e^{3x}.$$

or simply

$$0 = 2e^{3x},$$

which does not hold for any real x. This impossible situation tells us that there is no particular solution of the assumed form (4.40).

As noted, Equations (4.36) and (4.39) are almost the same, the only difference between them being the constant multiple of x in the exponents of their respective nonhomogeneous terms $2e^{4x}$ and $2e^{3x}$. The equation (4.36) involving $2e^{4x}$ had a particular solution of the assumed form Ae^{4x}, whereas Equation (4.39) involving $2e^{3x}$ did *not* have one of the assumed form Ae^{3x}. What is the difference in these two so apparently similar cases?

The answer to this is found by examining the solutions of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0 \tag{4.41}$$

which is the homogeneous equation corresponding to both (4.36) and (4.39). The auxiliary equation is $m^2 - 2m - 3 = 0$ with roots 3 and -1; and so

$$e^{3x} \quad \text{and} \quad e^{-x}$$

are (linearly independent) solutions of (4.41). This suggests that the failure to obtain a *solution of the form $y_p = Ae^{3x}$ for Equation (4.39) is due to the fact that the function e^{3x}* in this assumed solution is a solution of the homogeneous equation (4.41) corresponding to (4.39); and this is indeed the case. For, since Ae^{3x} satisfies the *homogeneous* equation (4.41), it reduces the common left member

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y$$

of both (4.41) and (4.39) to 0, *not* $2e^{3x}$, which a particular solution of Equation (4.39) would have to do.

Now that we have considered what caused the difficulty in attempting to obtain a particular solution of the form Ae^{3x} for (4.39), we naturally ask what form of solution should we seek? Recall that in the case of a double root m for an auxiliary equation, a solution linearly independent of the basic solution e^{mx} was xe^{mx}. While this in itself tells us nothing about the situation at hand, it might suggest that we seek a particular solution of (4.39) of the form

$$y_p = Axe^{3x}. \tag{4.42}$$

Differentiating (4.42), we obtain

$$y_p' = 3Axe^{3x} + Ae^{3x}, \qquad y_p'' = 9Axe^{3x} + 6Ae^{3x}.$$

Then substituting into (4.39), we obtain

$$(9Axe^{3x} + 6Ae^{3x}) - 2(3Axe^{3x} + Ae^{3x}) - 3Axe^{3x} = 2e^{3x}$$

or

$$(9A - 6A - 3A)xe^{3x} + 4Ae^{3x} = 2e^{3x}.$$

or simply

$$0xe^{3x} + 4Ae^{3x} = 2e^{3x}. \tag{4.43}$$

Since the (assumed) solution (4.42) is to satisfy the differential equation identically for *all* x on some real interval, the relation (4.43) must be an identity for all such x and hence the coefficients of e^{3x} on both sides of (4.43) must be respectively equal. Equating coefficients, we obtain the equation

$$4A = 2,$$

from which we determine the previously undetermined coefficient

$$A = \tfrac{1}{2}.$$

Substituting this back into (4.42), we obtain the particular solution

$$y_p = \tfrac{1}{2}xe^{3x}.$$

We summarize the results of this example. The differential equations

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x} \tag{4.36}$$

and

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{3x} \tag{4.39}$$

each have the same corresponding homogeneous equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0. \tag{4.41}$$

This homogeneous equation has linearly independent solutions

$$e^{3x} \quad \text{and} \quad e^{-x},$$

and so the complementary function of both (4.36) and (4.39) is

$$y_c = c_1 e^{3x} + c_2 e^{-x}.$$

The right member $2e^{4x}$ of (4.36) is *not* a solution of the corresponding homogeneous equation (4.41), and the attempted particular solution

$$y_p = Ae^{4x} \tag{4.37}$$

suggested by this right member did indeed lead to a particular solution of this assumed form, namely, $y_p = \frac{2}{5}e^{4x}$. On the other hand, the right member $2e^{3x}$ of (4.39) *is* a solution of the corresponding homogeneous equation (4.41) [with $c_1 = 2$ and $c_2 = 0$], and the attempted particular solution

$$y_p = Ae^{3x} \tag{4.40}$$

suggested by this right member *failed* to lead to a particular solution of this form. However, in this case, the revised attempted particular solution,

$$y_p = Axe^{3x}, \tag{4.42}$$

obtained from (4.40) by multiplying by x, led to a particular solution of this assumed form, namely, $y_p = \frac{1}{2}xe^{3x}$.

The general solutions of (4.36) and (4.39) are, respectively,

$$y = c_1 e^{3x} + c_2 e^{-x} + \tfrac{2}{5}e^{4x}$$

and

$$y = c_1 e^{3x} + c_2 e^{-x} + \tfrac{1}{2}xe^{3x}.$$

The preceding example illustrates a particular case of the method of undetermined coefficients. It suggests that in some cases the assumed particular solution y_p corresponding to a nonhomogeneous term in the differential equation is of the same type as that nonhomogeneous term, whereas in other cases the assumed y_p ought to be some sort of modification of that nonhomogeneous term. It turns out that this is essentially the case. We now proceed to present the method systematically.

B. The Method

We begin by introducing certain preliminary definitions.

DEFINITION

We shall call a function a UC function *if it is* either *(1) a function defined by one of the following:*

(*i*) x^n, *where n is a positive integer or zero,*
(*ii*) e^{ax}, *where a is a constant* $\neq 0$,
(*iii*) $\sin(bx + c)$, *where b and c are constants,* $b \neq 0$,
(*iv*) $\cos(bx + c)$, *where b and c are constants,* $b \neq 0$,

or *(2) a function defined as a finite product of two or more functions of these four types.*

▶ Example 4.20

Examples of UC functions of the four basic types (*i*), (*ii*), (*iii*), (*iv*) of the preceeding definition are those defined respectively by

$$x^3, \qquad e^{-2x}, \qquad \sin(3x/2), \qquad \cos(2x + \pi/4).$$

Examples of UC functions defined as finite products of two or more of these four basic types are those defined respectively by

$$x^2e^{3x}, \qquad x\cos 2x, \qquad e^{5x}\sin 3x,$$

$$\sin 2x \cos 3x, \qquad x^3e^{4x}\sin 5x.$$

The method of undetermined coefficients applies when the nonhomogeneous function F in the differential equation is a finite linear combination of UC functions.

Observe that given a UC function f, each successive derivative of f is either itself a constant multiple of a UC function or else a linear combination of UC functions.

DEFINITION

Consider a UC function f. The set of functions consisting of f itself and all linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations will be called the UC set *of f.*

► **Example 4.31**

The function f defined for all real x by $f(x) = x^3$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6 = 6 \cdot 1, \quad f^{(n)}(x) = 0 \quad \text{for} \quad n > 3.$$

The linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations are those given by

$$x^2, \quad x, \quad 1.$$

Thus the *UC set* of x^3 is the set $S = \{x^3, x^2, x, 1\}$.

► **Example 4.32**

The function f defined for all real x by $f(x) = \sin 2x$ is a UC function. Computing derivatives of f, we find

$$f'(x) = 2 \cos 2x, \quad f''(x) = -4 \sin 2x, \quad \ldots.$$

The only linearly independent UC function of which the successive derivatives of f are constant multiples or linear combinations is that given by $\cos 2x$. Thus the *UC set* of $\sin 2x$ is the set $S = \{\sin 2x, \cos 2x\}$.

These and similar examples of the four basic types of UC functions lead to the results listed as numbers 1, 2, and 3 of Table 4.1.

► **Example 4.33**

The function f defined for all real x by $f(x) = x^2 \sin x$ is the product of the two UC functions defined by x^2 and $\sin x$. Hence f is itself a UC function. Computing derivatives of f, we find

$$f'(x) = 2x \sin x + x^2 \cos x,$$

$$f''(x) = 2 \sin x + 4x \cos x - x^2 \sin x,$$

$$f'''(x) = 6 \cos x - 6x \sin x - x^2 \cos x, \quad \cdots.$$

No "new" types of functions will occur from further differentiation. Each derivative of f is a linear combination of certain of the six UC functions given by $x^2 \sin x$, $x^2 \cos x$,

TABLE 4.1

	UC function	*UC set*
1	x^n	$\{x^n, x^{n-1}, x^{n-2}, \ldots, x, 1\}$
2	e^{ax}	$\{e^{ax}\}$
3	$\sin(bx + c)$ or $\cos(bx + c)$	$\{\sin(bx + c), \cos(bx + c)\}$
4	$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1} e^{ax}, x^{n-2} e^{ax}, \ldots, xe^{ax}, e^{ax}\}$
5	$x^n \sin(bx + c)$ or $x^n \cos(bx + c)$	$\{x^n \sin(bx + c), x^n \cos(bx + c),$ $x^{n-1} \sin(bx + c), x^{n-1} \cos(bx + c),$ $\ldots, x \sin(bx + c), x \cos(bx + c),$ $\sin(bx + c), \cos(bx + c)\}$
6	$e^{ax} \sin(bx + c)$ or $e^{ax} \cos(bx + c)$	$\{e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$
7	$x^n e^{ax} \sin(bx + c)$ or $x^n e^{ax} \cos(bx + c)$	$\{x^n e^{ax} \sin(bx + c), x^n e^{ax} \cos(bx + c),$ $x^{n-1} e^{ax} \sin(bx + c), x^{n-1} e^{ax} \cos(bx + c), \ldots,$ $xe^{ax} \sin(bx + c), xe^{ax} \cos(bx + c),$ $e^{ax} \sin(bx + c), e^{ax} \cos(bx + c)\}$

$x \sin x$, $x \cos x$, $\sin x$, and $\cos x$. Thus the set

$$S = \{x^2 \sin x, x^2 \cos x, x \sin x, x \cos x, \sin x, \cos x\}$$

is the *UC set* of $x^2 \sin x$. Note carefully that x^2, x, and 1 are *not* members of this UC set.

Observe that the UC set of the product $x^2 \sin x$ is the set of all products obtained by multiplying the various members of the UC set $\{x^2, x, 1\}$ of x^2 by the various members of the UC set $\{\sin x, \cos x\}$ of $\sin x$. This observation illustrates the general situation regarding the UC set of a UC function defined as a finite product of two or more UC functions of the four basic types. In particular, suppose h is a UC function defined as the product fg of two basic UC functions f and g. Then the UC set of the product function h is the set of all the products obtained by multiplying the various members of the UC set of f by the various members of the UC set of g. Results of this type are listed as numbers 4, 5, and 6 of Table 4.1 and a specific illustration is presented in Example 4.34.

► Example 4.34

The function defined for all real x by $f(x) = x^3 \cos 2x$ is the product of the two UC functions defined by x^3 and $\cos 2x$. Using the result stated in the preceding paragraph, the UC set of this product $x^3 \cos 2x$ is the set of all products obtained by multiplying the various members of the UC set of x^3 by the various members of the UC set of $\cos 2x$. Using the definition of UC set or the appropriate numbers of Table 4.1, we find that the UC set of x^3 is

$$\{x^3, x^2, x, 1\}$$

and that of cos $2x$ is

$$\{\sin 2x, \cos 2x\}.$$

Thus the UC set of the product $x^3 \cos 2x$ is the set of all products of each of x^3, x^2, x, and 1 by each of $\sin 2x$ and $\cos 2x$, and so it is

$$\{x^3 \sin 2x, x^3 \cos 2x, x^2 \sin 2x, x^2 \cos 2x, x \sin 2x, x \cos 2x, \sin 2x, \cos 2x\}.$$

Observe that this can be found directly from Table 4.1, number 5, with $n = 3$, $b = 2$, and $c = 0$.

We now outline the method of undetermined coefficients for finding a particular integral y_p of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x),$$

where F is a finite linear combination

$$F = A_1 u_1 + A_2 u_2 + \cdots + A_m u_m$$

of UC functions $u_1, u_2, \ldots, u_m$, the A_i being known constants. Assuming the complementary function y_c has already been obtained, we proceed as follows:

1. For *each* of the UC functions

$$u_1, \ldots, u_m$$

of which F is a linear combination, form the corresponding UC set, thus obtaining the respective sets

$$S_1, S_2, \ldots, S_m.$$

2. Suppose that one of the UC sets so formed, say S_j, is identical with or completely included in another, say S_k. In this case, we omit the (identical or smaller) set S_j from further consideration (retaining the set S_k).

3. We now consider in turn each of the UC sets which still remain after Step 2. Suppose now that one of these UC sets, say S_l, includes one or more members which are solutions of the corresponding homogeneous differential equation. If this is the case, we multiply *each* member of S_l by the lowest positive integral power of x so that the resulting revised set will contain no members that are solutions of the corresponding homogeneous differential equation. We now replace S_l by this revised set, so obtained. Note that here we consider one UC set at a time and perform the indicated multiplication, if needed, only upon the members of the one UC set under consideration at the moment.

4. In general there now remains:
   (i) certain of the original UC sets, which were neither omitted in Step 2 nor needed revision in Step 3, and
   (ii) certain revised sets resulting from the needed revision in Step 3.

Now form a linear combination of *all* of the sets of these two categories, with unknown constant coefficients (*undetermined coefficients*).

5. Determine these unknown coefficients by substituting the linear combination formed in Step 4 into the differential equation and demanding that it identically satisfy the differential equation (that is, that it be a particular solution).

This outline of procedure at once covers all of the various special cases to which the method of undetermined coefficients applies, thereby freeing one from the need of considering separately each of these special cases.

Before going on to the illustrative examples of Part C following, let us look back and observe that we actually followed this procedure in solving the differential equations (4.36) and (4.39) of the Introductory Example 4.29. In each of those equations, the nonhomogeneous member consisted of a single term that was a constant multiple of a UC function; and in each case we followed the outline procedure step by step, as far as it applied.

For the differential equation (4.36), the UC function involved was e^{4x}; and we formed its UC set, which was simply $\{e^{4x}\}$ (Step 1). Step 2 obviously did not apply. Nor did Step 3, for as we noted later, e^{4x} was not a solution of the corresponding homogeneous equation (4.41). Thus we assumed $y_p = Ae^{4x}$ (Step 4) substituted in differential equation (4.36), and found A and hence y_p (Step 5).

For the differential equation (4.39), the UC function involved was e^{3x}; and we formed its UC set, which was simply $\{e^{3x}\}$ (Step 1). Step 2 did not apply here either. But Step 3 was very much needed, for e^{3x} *was* a solution of the corresponding homogeneous equation (4.41). Thus we applied Step 3 and multiplied e^{3x} in the UC set $\{e^{3x}\}$ by x, obtaining the revised UC set $\{xe^{3x}\}$, whose single member was *not* a solution of (4.41). Thus we assumed $y_p = Axe^{3x}$ (Step 4), substituted in the differential equation (4.39), and found A and hence y_p (Step 5).

The outline generalizes what the procedure for the differential equation of Introductory Example 4.29 suggested. Equation (4.39) of that example has already brought out the necessity for the revision described in Step 3 when it applies. We give here a brief illustration involving this critical step.

▶ Example 4.35

Consider the two equations

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^x \tag{4.44}$$

and

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^x \tag{4.45}$$

The UC set of x^2e^x is

$$S = \{x^2e^x, xe^x, e^x\}.$$

The homogeneous equation corresponding to (4.44) has linearly independent solutions e^x and e^{2x}, and so the complementary function of (4.44) is $y_c = c_1e^x + c_2e^{2x}$. Since member e^x of UC set S is a solution of the homogeneous equation corresponding to (4.44), we multiply each member of UC set S by the lowest positive integral power of x so that the resulting revised set will contain no members that are solutions of the homogeneous equation corresponding to (4.44). This turns out to be x itself; for the revised set

$$S' = \{x^3e^x, x^2e^x, xe^x\}$$

has no members that satisfy the homogeneous equation corresponding to (4.44).

The homogeneous equation corresponding to (4.45) has linearly independent solutions e^x and xe^x, and so the complementary function of (4.45) is $y_c = c_1 e^x + c_2 xe^x$. Since the two members e^x and xe^x of UC set S are solutions of the homogeneous equation corresponding to (4.45), we must modify S here also. But now x itself will not do, for we would get S', which still contains xe^x. Thus we must here multiply each member of S by x^2 to obtain the revised set

$$S'' = \{x^4e^x, x^3e^x, x^2e^x\},$$

which has no member that satisfies the homogeneous equation corresponding to (4.45).

C. Examples

A few illustrative examples, with reference to the above outline, should make the procedure clear. Our first example will be a simple one in which the situations of Steps 2 and 3 do not occur.

▶ Example 4.36

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10 \sin x.$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

and the complementary function is

$$y_c = c_1 e^{3x} + c_2 e^{-x}.$$

The nonhomogenous term is the linear combination $2e^x - 10 \sin x$ of the two UC functions given by e^x and $\sin x$.

1. Form the UC set for each of these two functions. We find

$$S_1 = \{e^x\},$$
$$S_2 = \{\sin x, \cos x\}.$$

2. Note that neither of these sets is identical with nor included in the other; hence both are retained.

3. Furthermore, by examining the complementary function, we see that none of the functions e^x, $\sin x$, $\cos x$ in either of these sets is a solution of the corresponding homogeneous equation. Hence neither set needs to be revised.

4. Thus the original sets S_1 and S_2 remain intact in this problem, and we form the linear combination

$$Ae^x + B \sin x + C \cos x$$

of the three elements e^x, $\sin x$, $\cos x$ of S_1 and S_2, with the undetermined coefficients A, B, C.

5. We determine these unknown coefficients by substituting the linear combination formed in Step 4 into the differential equation and demanding that it satisfy the differential equation identically. That is, we take

$$y_p = Ae^x + B \sin x + C \cos x$$

as a particular solution. Then

$$y'_p = Ae^x + B \cos x - C \sin x,$$
$$y''_p = Ae^x - B \sin x - C \cos x.$$

Actually substituting, we find

$$(Ae^x - B \sin x - C \cos x) - 2(Ae^x + B \cos x - C \sin x) \\ - 3(Ae^x + B \sin x + C \cos x) = 2e^x - 10 \sin x$$

or

$$-4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10 \sin x.$$

Since the solution is to satisfy the differential equation identically for *all* x on some real interval, this relation must be an identity for all such x and hence the coefficients of like terms on both sides must be respectively equal. Equating coefficients of these like terms, we obtain the equations

$$-4A = 2, \qquad -4B + 2C = -10, \qquad -4C - 2B = 0.$$

From these equations, we find that

$$A = -\tfrac{1}{2}, \qquad B = 2, \qquad C = -1,$$

and hence we obtain the particular integral

$$y_p = -\tfrac{1}{2}e^x + 2 \sin x - \cos x.$$

Thus the general solution of the differential equation under consideration is

$$y = y_c + y_p = c_1 e^{3x} + c_2 e^{-x} - \tfrac{1}{2}e^x + 2 \sin x - \cos x.$$

► **Example 4.37**

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}.$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

and the complementary function is

$$y_c = c_1 e^x + c_2 e^{2x}.$$

The nonhomogeneous term is the linear combination

$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

of the four UC functions given by x^2, e^x, xe^x, and e^{3x}.

1. Form the UC set for each of these functions. We have

$$S_1 = \{x^2, x, 1\},$$
$$S_2 = \{e^x\},$$
$$S_3 = \{xe^x, e^x\},$$
$$S_4 = \{e^{3x}\}.$$

2. We note that S_2 is completely included in S_3, so S_2 is omitted from further consideration, leaving the three sets

$$S_1 = \{x^2, x, 1\} \qquad S_3 = \{xe^x, e^x\}, \qquad S_4 = \{e^{3x}\}.$$

3. We now observe that $S_3 = \{xe^x, e^x\}$ includes e^x, which is included in the complementary function and so is a solution of the corresponding homogeneous differential equation. Thus we multiply *each* member of S_3 by x to obtain the revised family

$$S_3' = \{x^2e^x, xe^x\},$$

which contains no members that are solutions of the corresponding homogeneous equation.

4. Thus there remain the original UC sets

$$S_1 = \{x^2, x, 1\}$$

and

$$S_4 = \{e^{3x}\}$$

and the revised set

$$S_3' = \{x^2e^x, xe^x\}.$$

These contain the six elements

$$x^2, \quad x, \quad 1, \quad e^{3x}, \quad x^2e^x, \quad xe^x.$$

We form the linear combination

$$Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x$$

of these six elements.

5. Thus we take as our particular solution,

$$y_p = Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x.$$

From this, we have

$$y_p' = 2Ax + B + 3De^{3x} + Ex^2e^x + 2Exe^x + Fxe^x + Fe^x,$$
$$y_p'' = 2A + 9De^{3x} + Ex^2e^x + 4Exe^x + 2Ee^x + Fxe^x + 2Fe^x.$$

We substitute y_p, y_p', y_p'' into the differential equation for y, dy/dx, d^2y/dx^2, respectively, to obtain:

$$\begin{aligned} 2A + 9De^{3x} + Ex^2e^x + (4E + F)xe^x + (2E + 2F)e^x \\ - 3[2Ax + B + 3De^{3x} + Ex^2e^x + (2E + F)xe^x + Fe^x] \\ + 2(Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x) \\ = 2x^2 + e^x + 2xe^x + 4e^{3x}, \end{aligned}$$

or

$$(2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 + 2De^{3x} + (-2E)xe^x + (2E - F)e^x$$
$$= 2x^2 + e^x + 2xe^x + 4e^{3x}.$$

Equating coefficients of like terms, we have:

$$2A - 3B + 2C = 0,$$
$$2B - 6A = 0,$$
$$2A = 2,$$
$$2D = 4,$$
$$-2E = 2,$$
$$2E - F = 1.$$

From this $A = 1, B = 3, C = \frac{7}{2}, D = 2, E = -1, F = -3$, and so the particular integral is

$$y_p = x^2 + 3x + \tfrac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$

The general solution is therefore

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + x^2 + 3x + \tfrac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$

▶ **Example 4.38**

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4\sin x - 2\cos x.$$

The corresponding homogeneous equation is

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 0,$$

and the complementary function is

$$y_c = c_1 + c_2x + c_3\sin x + c_4\cos x.$$

The nonhomogeneous term is the linear combination

$$3x^2 + 4\sin x - 2\cos x$$

of the three UC functions given by

$$x^2, \quad \sin x, \quad \text{and} \quad \cos x.$$

1. Form the UC set for each of these three functions. These sets are, respectively,

$$S_1 = \{x^2, x, 1\},$$
$$S_2 = \{\sin x, \cos x\},$$
$$S_3 = \{\cos x, \sin x\}.$$

2. Observe that S_2 and S_3 are identical and so we retain only one of them, leaving the two sets

$$S_1 = \{x^2, x, 1\}, \qquad S_2 = \{\sin x, \cos x\}.$$

3. Now observe that $S_1 = \{x^2, x, 1\}$ includes 1 and x, which, as the complementary function shows, are both solutions of the corresponding homogeneous differential equation. Thus we multiply each member of the set S_1 by x^2 to obtain the revised set

$$S_1' = \{x^4, x^3, x^2\},$$

none of whose members are solutions of the homogeneous differential equation. We observe that multiplication by x instead of x^2 would not be sufficient, since the resulting set would be $\{x^3, x^2, x\}$, which still includes the homogeneous solution x. Turning to the set S_2, observe that both of its members, $\sin x$ and $\cos x$, are also solutions of the homogeneous differential equation. Hence we replace S_2 by the revised set

$$S_2' = \{x \sin x, x \cos x\}.$$

4. None of the original UC sets remain here. They have been replaced by the revised sets S_1' and S_2' containing the five elements

$$x^4, \quad x^3, \quad x^2, \quad x \sin x, \quad x \cos x.$$

We form a linear combination of these,

$$Ax^4 + Bx^3 + Cx^2 + Dx \sin x + Ex \cos x,$$

with undetermined coefficients A, B, C, D, E.

5. We now take this as our particular solution

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx \sin x + Ex \cos x.$$

Then

$$y_p' = 4Ax^3 + 3Bx^2 + 2Cx + Dx \cos x + D \sin x - Ex \sin x + E \cos x,$$

$$y_p'' = 12Ax^2 + 6Bx + 2C - Dx \sin x + 2D \cos x - Ex \cos x - 2E \sin x,$$

$$y_p''' = 24Ax + 6B - Dx \cos x - 3D \sin x + Ex \sin x - 3E \cos x,$$

$$y_p^{(iv)} = 24A + Dx \sin x - 4D \cos x + Ex \cos x + 4E \sin x.$$

Substituting into the differential equation, we obtain

$$24A + Dx \sin x - 4D \cos x + Ex \cos x + 4E \sin x + 12Ax^2 + 6Bx + 2C$$
$$- Dx \sin x + 2D \cos x - Ex \cos x - 2E \sin x$$
$$= 3x^2 + 4 \sin x - 2 \cos x.$$

Equating coefficients, we find

$$24A + 2C = 0$$
$$6B = 0$$
$$12A = 3$$
$$-2D = -2$$
$$2E = 4.$$

Hence $A = \frac{1}{4}$, $B = 0$, $C = -3$, $D = 1$, $E = 2$, and the particular integral is

$$y_p = \tfrac{1}{4}x^4 - 3x^2 + x \sin x + 2x \cos x.$$

The general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 + c_2 x + c_3 \sin x + c_4 \cos x + \tfrac{1}{4}x^4 - 3x^2 + x \sin x + 2x \cos x. \end{aligned}$$

▶ Example 4.39 An Initial-Value Problem

We close this section by applying our results to the solution of the initial-value problem

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10 \sin x, \tag{4.46}$$

$$y(0) = 2, \tag{4.47}$$

$$y'(0) = 4. \tag{4.48}$$

By Theorem 4.1, this problem has a unique solution, defined for all x, $-\infty < x < \infty$; let us proceed to find it. In Example 4.36 we found that the general solution of the differential equation (4.46) is

$$y = c_1 e^{3x} + c_2 e^{-x} - \tfrac{1}{2}e^x + 2 \sin x - \cos x. \tag{4.49}$$

From this, we have

$$\frac{dy}{dx} = 3c_1 e^{3x} - c_2 e^{-x} - \tfrac{1}{2}e^x + 2 \cos x + \sin x. \tag{4.50}$$

Applying the initial conditions (4.47) and (4.48) to Equations (4.49) and (4.50), respectively, we have

$$\begin{aligned} 2 &= c_1 e^0 + c_2 e^0 - \tfrac{1}{2}e^0 + 2 \sin 0 - \cos 0, \\ 4 &= 3c_1 e^0 - c_2 e^0 - \tfrac{1}{2}e^0 + 2 \cos 0 + \sin 0. \end{aligned}$$

These equations simplify at once to the following:

$$c_1 + c_2 = \tfrac{7}{2}, \qquad 3c_1 - c_2 = \tfrac{5}{2}.$$

From these two equations we obtain

$$c_1 = \tfrac{3}{2}, \qquad c_2 = 2.$$

Substituting these values for c_1 and c_2 into Equation (4.49) we obtain the unique solution of the given initial-value problem in the form

$$y = \tfrac{3}{2}e^{3x} + 2e^{-x} - \tfrac{1}{2}e^x + 2 \sin x - \cos x.$$

Exercises

Find the general solution of each of the differential equations in Exercises 1–24.

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x^2.$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 4e^{2x} - 21e^{-3x}.$

3. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 6\sin 2x + 7\cos 2x.$

4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 10\sin 4x.$

5. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = \cos 4x.$

6. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 16x - 12e^{2x}.$

7. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 2e^x + 10e^{5x}.$

8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 5xe^{-2x}.$

9. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = -18x^2 + 1.$

10. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = 8xe^{-2x}.$

11. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 5y = 5\sin 2x + 10x^2 + 3x + 7.$

12. $4\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 3y = 3x^3 - 8x.$

13. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 10e^{2x} - 18e^{3x} - 6x - 11.$

14. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 6e^{-2x} + 3e^x - 4x^2.$

15. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 4e^x - 18e^{-x}.$

16. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 9e^{2x} - 8e^{3x}.$

17. $\frac{d^3y}{dx^3} + \frac{dy}{dx} = 2x^2 + 4\sin x.$

18. $\frac{d^4y}{dx^4} - 3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 3e^{-x} + 6e^{2x} - 6x.$

19. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = xe^x - 4e^{2x} + 6e^{4x}.$

20. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 3x^2e^x - 7e^x.$

21. $\dfrac{d^2y}{dx^2} + y = x \sin x.$

22. $\dfrac{d^2y}{dx^2} + 4y = 12x^2 - 16x \cos 2x.$

23. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} = 18x^2 + 16xe^x + 4e^{3x} - 9.$

24. $\dfrac{d^4y}{dx^4} - 5\dfrac{d^3y}{dx^3} + 7\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 5 \sin x - 12 \sin 2x.$

Solve the initial-value problems in Exercises 25–40.

25. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 9x^2 + 4, \quad y(0) = 6, \quad y'(0) = 8.$

26. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 16x + 20e^x, \quad y(0) = 0, \quad y'(0) = 3.$

27. $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 15y = 9xe^{2x}, \quad y(0) = 5, \quad y'(0) = 10.$

28. $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} + 10y = 4xe^{-3x}, \quad y(0) = 0, \quad y'(0) = -1.$

29. $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 8e^{-2x}, \quad y(0) = 2, \quad y'(0) = 0.$

30. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 27e^{-6x}, \quad y(0) = -2, \quad y'(0) = 0.$

31. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 18e^{-2x}, \quad y(0) = 0, \quad y'(0) = 4.$

32. $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 29y = 8e^{5x}, \quad y(0) = 0, \quad y'(0) = 8.$

33. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 8 \sin 3x, \quad y(0) = 1, \quad y'(0) = 2.$

34. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 8e^{2x} - 5e^{3x}, \quad y(0) = 3, \quad y'(0) = 5.$

35. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 2xe^{2x} + 6e^x, \quad y(0) = 1, \quad y'(0) = 0.$

36. $\dfrac{d^2y}{dx^2} - y = 3x^2e^x, \quad y(0) = 1, \quad y'(0) = 2.$

37. $\dfrac{d^2y}{dx^2} + y = 3x^2 - 4 \sin x, \quad y(0) = 0, \quad y'(0) = 1.$

38. $\dfrac{d^2y}{dx^2} + 4y = 8 \sin 2x, \quad y(0) = 6, \quad y'(0) = 8.$

39. $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 6y = 3xe^x + 2e^x - \sin x,$

$$y(0) = \frac{33}{40}, \qquad y'(0) = 0, \qquad y''(0) = 0.$$

40. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 9\dfrac{dy}{dx} - 4y = 8x^2 + 3 - 6e^{2x},$

$$y(0) = 1, \qquad y'(0) = 7, \qquad y''(0) = 10.$$

For each of the differential equations in Exercises 41–54 *set up* the correct linear combination of functions with undetermined literal coefficients to use in finding a particular integral by the method of undetermined coefficients. (Do not actually find the particular integrals.)

41. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = x^3 + x + e^{-2x}.$

42. $\dfrac{d^2y}{dx^2} + 9y = e^{3x} + e^{-3x} + e^{3x}\sin 3x.$

43. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = e^{-2x}(1 + \cos x).$

44. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = x^4e^x + x^3e^{2x} + x^2e^{3x}.$

45. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 13y = xe^{-3x}\sin 2x + x^2e^{-2x}\sin 3x.$

46. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = x^2e^x + 3xe^{2x} + 5x^2.$

47. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} - 8y = xe^{2x} + x^2e^{3x}.$

48. $\dfrac{d^4y}{dx^4} + 3\dfrac{d^3y}{dx^3} + 4\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = x^2e^{-x} + 3e^{-x/2}\cos\dfrac{\sqrt{3}}{2}x.$

49. $\dfrac{d^4y}{dx^4} - 16y = x^2\sin 2x + x^4e^{2x}.$

50. $\dfrac{d^6y}{dx^6} + 2\dfrac{d^5y}{dx^5} + 5\dfrac{d^4y}{dx^4} = x^3 + x^2e^{-x} + e^{-x}\sin 2x.$

51. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2\cos x.$

52. $\dfrac{d^4y}{dx^4} + 16y = xe^{\sqrt{2}x}\sin\sqrt{2}x + e^{-\sqrt{2}x}\cos\sqrt{2}x.$

53. $\dfrac{d^4y}{dx^4} + 3\dfrac{d^2y}{dx^2} - 4y = \cos^2 x - \cosh x.$

54. $\dfrac{d^4y}{dx^4} + 10\dfrac{d^2y}{dx^2} + 9y = \sin x \sin 2x.$

4.4 VARIATION OF PARAMETERS

A. The Method

While the process of carrying out the method of undetermined coefficients is actually quite straightforward (involving only techniques of college algebra and differentiation), the method applies in general to a rather small class of problems. For example, it would not apply to the apparently simply equation

$$\frac{d^2y}{dx^2} + y = \tan x.$$

We thus seek a method of finding a particular integral that applies in all cases (including variable coefficients) in which the complementary function is known. Such a method is the method of *variation of parameters*, which we now consider.

We shall develop this method in connection with the general second-order linear differential equation with variable coefficients

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x). \tag{4.51}$$

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \tag{4.52}$$

Then the complementary function of Equation (4.51) is

$$c_1y_1(x) + c_2y_2(x),$$

where y_1 and y_2 are linearly independent solutions of (4.52) and c_1 and c_2 are arbitrary constants. The procedure in the method of variation of parameters is to replace the arbitrary constants c_1 and c_2 in the complementary function by respective *functions* v_1 and v_2 which will be determined so that the resulting function, which is defined by

$$v_1(x)y_1(x) + v_2(x)y_2(x), \tag{4.53}$$

will be a particular integral of Equation (4.51) (hence the name, *variation* of parameters).

We have at our disposal the *two functions* v_1 and v_2 with which to satisfy the *one condition* that (4.53) be a solution of (4.51). Since we have *two* functions but only *one* condition on them, we are thus free to impose a second condition, provided this second condition does not violate the first one. We shall see when and how to impose this additional condition as we proceed.

We thus assume a solution of the form (4.53) and write

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x). \tag{4.54}$$

Differentiating (4.54), we have

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x) + v_1'(x)y_1(x) + v_2'(x)y_2(x), \tag{4.55}$$

where we use primes to denote differentiations. At this point we impose the aforementioned second condition; we simplify y_p' by demanding that

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0. \tag{4.56}$$

With this condition imposed, (4.55) reduces to

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x). \tag{4.57}$$

Now differentiating (4.57), we obtain

$$y_p''(x) = v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x). \tag{4.58}$$

We now impose the basic condition that (4.54) be a solution of Equation (4.51). Thus we *substitute (4.54), (4.57), and (4.58) for y, dy/dx, and d^2y/dx^2*, respectively, in Equation (4.51) and obtain the identity

$$\begin{aligned} &a_0(x)[v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] \\ &\quad + a_1(x)[v_1(x)y_1'(x) + v_2(x)y_2'(x)] + a_2(x)[v_1(x)y_1(x) + v_2(x)y_2(x)] = F(x). \end{aligned}$$

This can be written as

$$\begin{aligned} &v_1(x)[a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x)] \\ &\quad + v_2(x)[a_0(x)y_2''(x) + a_1(x)y_2'(x) + a_2(x)y_2(x)] \\ &\quad\quad + a_0(x)[v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = F(x). \end{aligned} \tag{4.59}$$

Since y_1 and y_2 are solutions of the corresponding homogeneous differential equation (4.52), the expressions in the first two brackets in (4.59) are identically zero. This leaves merely

$$v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = \frac{F(x)}{a_0(x)}. \tag{4.60}$$

This is actually what the basic condition demands. Thus the two imposed conditions require that the functions v_1 and v_2 be chosen such that the system of equations

$$\begin{aligned} y_1(x)v_1'(x) + y_2(x)v_2'(x) &= 0, \\ y_1'(x)v_1'(x) + y_2'(x)v_2'(x) &= \frac{F(x)}{a_0(x)}, \end{aligned} \tag{4.61}$$

is satisfied. The determinant of coefficients of this system is precisely

$$W[y_1(x), y_2(x)] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous differential equation (4.52), we know that $W[y_1(x), y_2(x)] \neq 0$. Hence the system (4.61) has a unique solution. Actually solving this system, we obtain

$$v_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ \dfrac{F(x)}{a_0(x)} & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = -\frac{F(x)y_2(x)}{a_0(x)W[y_1(x), y_2(x)]},$$

$$v_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \dfrac{F(x)}{a_0(x)} \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = \frac{F(x)y_1(x)}{a_0(x)W[y_1(x), y_2(x)]}.$$

Thus we obtain the functions v_1 and v_2 defined by

$$v_1(x) = -\int^x \frac{F(t)y_2(t)\,dt}{a_0(t)W[y_1(t), y_2(t)]},$$

$$v_2(x) = \int^x \frac{F(t)y_1(t)\,dt}{a_0(t)W[y_1(t), y_2(t)]}. \tag{4.62}$$

Therefore a particular integral y_p of Equation (4.51) is defined by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

where v_1 and v_2 are defined by (4.62).

B. Examples

▶ Example 4.40

Consider the differential equation

$$\frac{d^2y}{dx^2} + y = \tan x. \tag{4.63}$$

The complementary function is defined by

$$y_c(x) = c_1 \sin x + c_2 \cos x.$$

We assume

$$y_p(x) = v_1(x)\sin x + v_2(x)\cos x, \tag{4.64}$$

where the functions v_1 and v_2 will be determined such that this is a particular integral of the differential equation (4.63). Then

$$y_p'(x) = v_1(x)\cos x - v_2(x)\sin x + v_1'(x)\sin x + v_2'(x)\cos x.$$

We impose the condition

$$v_1'(x)\sin x + v_2'(x)\cos x = 0, \tag{4.65}$$

leaving

$$y_p'(x) = v_1(x)\cos x - v_2(x)\sin x.$$

From this

$$y_p''(x) = -v_1(x)\sin x - v_2(x)\cos x + v_1'(x)\cos x - v_2'(x)\sin x. \tag{4.66}$$

Substituting (4.64) and (4.66) into (4.63) we obtain

$$v_1'(x)\cos x - v_2'(x)\sin x = \tan x. \tag{4.67}$$

Thus we have the two equations (4.65) and (4.67) from which to determine $v_1'(x)$, $v_2'(x)$:

$$v_1'(x)\sin x + v_2'(x)\cos x = 0,$$
$$v_1'(x)\cos x - v_2'(x)\sin x = \tan x.$$

Solving we find:

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{-\cos x \tan x}{-1} = \sin x,$$

$$v_2'(x) = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \tan x}{-1} = \frac{-\sin^2 x}{\cos x}$$
$$= \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x.$$

Integrating we find:

$$v_1(x) = -\cos x + c_3, \qquad v_2(x) = \sin x - \ln|\sec x + \tan x| + c_4. \tag{4.68}$$

Substituting (4.68) into (4.64) we have

$$\begin{aligned} y_p(x) &= (-\cos x + c_3)\sin x + (\sin x - \ln|\sec x + \tan x| + c_4)\cos x \\ &= -\sin x \cos x + c_3 \sin x + \sin x \cos x \\ &\quad -\ln|\sec x + \tan x|(\cos x) + c_4 \cos x \\ &= c_3 \sin x + c_4 \cos x - (\cos x)(\ln|\sec x + \tan x|). \end{aligned}$$

Since a particular integral is a solution free of arbitrary constants, we may assign any particular values A and B to c_3 and c_4, respectively, and the result will be the particular integral

$$A \sin x + B \cos x - (\cos x)(\ln|\sec x + \tan x|).$$

Thus $y = y_c + y_p$ becomes

$$y = c_1 \sin x + c_2 \cos x + A \sin x + B \cos x - (\cos x)(\ln|\sec x + \tan x|),$$

which we may write as

$$y = C_1 \sin x + C_2 \cos x - (\cos x)(\ln|\sec x + \tan x|),$$

where $C_1 = c_1 + A$, $C_2 = c_2 + B$.

Thus we see that we might as well have chosen the constants c_3 and c_4 both equal to 0 in (4.68), for essentially the same result,

$$y = c_1 \sin x + c_2 \cos x - (\cos x)(\ln|\sec x + \tan x|),$$

would have been obtained. This is the general solution of the differential equation (4.63).

The method of variation of parameters extends to higher-order linear equations. We now illustrate the extension to a third-order equation in Example 4.41, although we hasten to point out that the equation of this example can be solved more readily by the method of undetermined coefficients.

▶ Example 4.41

Consider the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^x. \tag{4.69}$$

The complementary function is

$$y_c(x) = c_1e^x + c_2e^{2x} + c_3e^{3x}.$$

We assume as a particular integral

$$y_p(x) = v_1(x)e^x + v_2(x)e^{2x} + v_3(x)e^{3x}. \tag{4.70}$$

Since we have *three* functions v_1, v_2, v_3 at our disposal in this case, we can apply three conditions. We have:

$$y_p'(x) = v_1(x)e^x + 2v_2(x)e^{2x} + 3v_3(x)e^{3x} + v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x}.$$

Proceeding in a manner analogous to that of the second-order case, we impose the condition

$$v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x} = 0, \tag{4.71}$$

leaving

$$y_p'(x) = v_1(x)e^x + 2v_2(x)e^{2x} + 3v_3(x)e^{3x}. \tag{4.72}$$

Then

$$y_p''(x) = v_1(x)e^x + 4v_2(x)e^{2x} + 9v_3(x)e^{3x} + v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x}.$$

We now impose the condition

$$v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x} = 0, \tag{4.73}$$

leaving

$$y_p''(x) = v_1(x)e^x + 4v_2(x)e^{2x} + 9v_3(x)e^{3x}. \tag{4.74}$$

From this,

$$y_p'''(x) = v_1(x)e^x + 8v_2(x)e^{2x} + 27v_3(x)e^{3x} + v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x}. \tag{4.75}$$

We substitute (4.70), (4.72), (4.74), and (4.75) into the differential equation (4.69), obtaining:

$$\begin{aligned} &v_1(x)e^x + 8v_2(x)e^{2x} + 27v_3(x)e^{3x} + v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x} \\ &\quad - 6v_1(x)e^x - 24v_2(x)e^{2x} - 54v_3(x)e^{3x} + 11v_1(x)e^x + 22v_2(x)e^{2x} + 33v_3(x)e^{3x} \\ &\qquad - 6v_1(x)e^x - 6v_2(x)e^{2x} - 6v_3(x)e^{3x} = e^x \end{aligned}$$

or

$$v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x} = e^x. \tag{4.76}$$

Thus we have the three equations (4.71), (4.73), (4.76) from which to determine $v_1'(x), v_2'(x), v_3'(x)$:

$$v_1'(x)e^x + v_2'(x)e^{2x} + v_3'(x)e^{3x} = 0,$$
$$v_1'(x)e^x + 2v_2'(x)e^{2x} + 3v_3'(x)e^{3x} = 0,$$
$$v_1'(x)e^x + 4v_2'(x)e^{2x} + 9v_3'(x)e^{3x} = e^x.$$

Solving, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{e^{6x}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}{e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}} = \frac{1}{2},$$

$$v_2'(x) = \frac{\begin{vmatrix} e^x & 0 & e^{3x} \\ e^x & 0 & 3e^{3x} \\ e^x & e^x & 9e^{3x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{-e^{5x}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{2e^{6x}} = -e^{-x},$$

$$v_3'(x) = \frac{\begin{vmatrix} e^x & e^{2x} & 0 \\ e^x & 2e^{2x} & 0 \\ e^x & 4e^{2x} & e^x \end{vmatrix}}{\begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}} = \frac{e^{4x}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{2e^{6x}} = \frac{1}{2}e^{-2x}.$$

We now integrate, choosing all the constants of integration to be zero (as the previous example showed was possible). We find:

$$v_1(x) = \tfrac{1}{2}x, \qquad v_2(x) = e^{-x}, \qquad v_3(x) = -\tfrac{1}{4}e^{-2x}.$$

Thus

$$y_p(x) = \tfrac{1}{2}xe^x + e^{-x}e^{2x} - \tfrac{1}{4}e^{-2x}e^{3x} = \tfrac{1}{2}xe^x + \tfrac{3}{4}e^x.$$

Thus the general solution of Equation (4.53) is

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} + \tfrac{1}{2}xe^x + \tfrac{3}{4}e^x$$

or

$$y = c_1'e^x + c_2e^{2x} + c_3e^{3x} + \tfrac{1}{2}xe^x,$$

where $c_1' = c_1 + \frac{3}{4}$.

In Examples 4.40 and 4.41 the coefficients in the differential equation were constants. The general discussion at the beginning of this section shows that the method applies equally well to linear differential equations with variable coefficients, once the complementary function y_c is known. We now illustrate its application to such an equation in Example 4.42.

▶ Example 4.42

Consider the differential equation

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2 + 1)^2. \tag{4.77}$$

In Example 4.16 we solved the corresponding homogeneous equation

$$(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

From the results of that example, we see that the complementary function of equation (4.77) is

$$y_c(x) = c_1 x + c_2(x^2 - 1).$$

To find a particular integral of Equation (4.77), we therefore let

$$y_p(x) = v_1(x)x + v_2(x)(x^2 - 1). \tag{4.78}$$

Then

$$y_p'(x) = v_1(x) \cdot 1 + v_2(x) \cdot 2x + v_1'(x)x + v_2'(x)(x^2 - 1).$$

We impose the condition

$$v_1'(x)x + v_2'(x)(x^2 - 1) = 0, \tag{4.79}$$

leaving

$$y_p'(x) = v_1(x) \cdot 1 + v_2(x) \cdot 2x. \tag{4.80}$$

From this, we find

$$y_p''(x) = v_1'(x) + 2v_2(x) + v_2'(x) \cdot 2x. \tag{4.81}$$

Substituting (4.78), (4.80), and (4.81) into (4.77) we obtain

$$(x^2 + 1)[v_1'(x) + 2v_2(x) + 2xv_2'(x)] - 2x[v_1(x) + 2xv_2(x)] + 2[v_1(x)x + v_2(x)(x^2 - 1)] = 6(x^2 + 1)^2$$

or

$$(x^2 + 1)[v_1'(x) + 2xv_2'(x)] = 6(x^2 + 1)^2. \tag{4.82}$$

Thus we have the two equations (4.79) and (4.82) from which to determine $v_1'(x)$ and $v_2'(x)$; that is, $v_1'(x)$ and $v_2'(x)$ satisfy the system

$$v_1'(x)x + v_2'(x)[x^2 - 1] = 0,$$
$$v_1'(x) + v_2'(x)[2x] = 6(x^2 + 1).$$

Solving this system, we find

$$v_1'(x) = \frac{\begin{vmatrix} 0 & x^2 - 1 \\ 6(x^2+1) & 2x \end{vmatrix}}{\begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix}} = \frac{-6(x^2+1)(x^2-1)}{x^2+1} = -6(x^2-1),$$

$$v_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & 6(x^2+1) \end{vmatrix}}{\begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix}} = \frac{6x(x^2+1)}{x^2+1} = 6x.$$

Integrating, we obtain

$$v_1(x) = -2x^3 + 6x, \qquad v_2(x) = 3x^2, \tag{4.83}$$

where we have chosen both constants of integration to be zero. Substituting (4.83) into (4.78), we have

$$\begin{aligned} y_p(x) &= (-2x^3 + 6x)x + 3x^2(x^2 - 1) \\ &= x^4 + 3x^2. \end{aligned}$$

Therefore the general solution of Equation (4.77) may be expressed in the form

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 x + c_2(x^2 - 1) + x^4 + 3x^2. \end{aligned}$$

Exercises

Find the general solution of each of the differential equations in Exercises 1–18.

1. $\frac{d^2y}{dx^2} + y = \cot x.$
2. $\frac{d^2y}{dx^2} + y = \tan^2 x.$
3. $\frac{d^2y}{dx^2} + y = \sec x.$
4. $\frac{d^2y}{dx^2} + y = \sec^3 x.$
5. $\frac{d^2y}{dx^2} + 4y = \sec^2 2x.$
6. $\frac{d^2y}{dx^2} + y = \tan x \sec x.$
7. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = e^{-2x} \sec x.$
8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^x \tan 2x.$
9. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \frac{e^{-3x}}{x^3}.$
10. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \ln x \; (x > 0).$

DEFINITION

Let $f_1, f_2, \ldots, f_n$ *be n real functions each of which has an* $(n-1)$*st derivative on a real interval* $a \le x \le b$. *The determinant*

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

in which primes denote derivatives, is called the Wronskian *of these n functions. We observe that $W(f_1, f_2, \ldots, f_n)$ is itself a real function defined on $a \le x \le b$. Its value at x is denoted by $W(f_1, f_2, \ldots, f_n)(x)$ or by $W[f_1(x), f_2(x), \ldots, f_n(x)]$.*

THEOREM 4.4

The n solutions $f_1, f_2, \ldots, f_n$ of the nth-order homogeneous linear differential equation (4.2) are linearly independent on $a \le x \le b$ if and only if the Wronskian of $f_1, f_2, \ldots, f_n$ is different from zero for some x on the interval $a \le x \le b$.

We have further:

THEOREM 4.5

The Wronskian of n solutions $f_1, f_2, \ldots, f_n$ of (4.2) is either identically zero on $a \le x \le b$ or else is never zero on $a \le x \le b$.

Thus if we can find n solutions of (4.2), we can apply the Theorems 4.4. and 4.5 to determine whether or not they are linearly independent. If they are linearly independent, then we can form the general solution as a linear combination of these n linearly independent solutions.

In the case of the general *second*-order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \tag{4.4}$$

the Wronskian of two solutions f_1 and f_2 is the second-order determinant

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f'_1 & f'_2 \end{vmatrix} = f_1 f'_2 - f'_1 f_2.$$

By Theorem 4.4, two solutions f_1 and f_2 of (4.4) are linearly independent on $a \le x \le b$ if and only if their Wronskian is different from zero for some x on $a \le x \le b$; and by Theorem 4.5, this Wronskian is either always zero or never zero on $a \le x \le b$. Thus if $W[f_1(x), f_2(x)] \ne 0$ on $a \le x \le b$, solutions f_1 and f_2 of (4.4) are linearly independent on $a \le x \le b$ and the general solution of (4.4) can be written as the linear combination

$$c_1 f_1(x) + c_2 f_2(x),$$

where c_1 and c_2 are arbitrary constants.

▶ Example 4.14

We apply Theorem 4.4 to show that the solutions $\sin x$ and $\cos x$ of

$$\frac{d^2y}{dx^2} + y = 0$$

are linearly independent. We find that

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

for all real x. Thus, since $W(\sin x, \cos x) \neq 0$ for all real x, we conclude that $\sin x$ and $\cos x$ are indeed linearly independent solutions of the given differential equation on every real interval.

► **Example 4.15**

The solutions e^x, e^{-x}, and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

are linearly independent on every real interval, for

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0$$

for all real x.

Exercises

1. Theorem 4.1 applies to one of the following problems but not to the other. Determine to which of the problems the theorem applies and state precisely the conclusion which can be drawn in this case. Explain why the theorem does not apply to the remaining problem.

   (a) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^x, \qquad y(0) = 5, \qquad y'(0) = 7.$

   (b) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^x, \qquad y(0) = 5, \qquad y'(1) = 7.$

2. Answer orally: What is the solution of the following initial-value problem? Why?

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0, \qquad y(1) = 0, \qquad y'(1) = 0.$$

3. Prove Theorem 4.2 for the case $m = n = 2$. That is, prove that if $f_1(x)$ and $f_2(x)$ are two solutions of

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0,$$

   then $c_1 f_1(x) + c_2 f_2(x)$ *is also a solution of this equation, where* c_1 *and* c_2 *are* arbitrary constants.

4. Consider the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0. \tag{A}$$

(a) Show that each of the functions e^x and e^{3x} is a solution of differential equation (A) on the interval $a \le x \le b$, where a and b are arbitrary real numbers such that $a < b$.

(b) What theorem enables us to conclude at once that each of the functions

$$5e^x + 2e^{3x}, \qquad 6e^x - 4e^{3x}, \quad \text{and} \quad -7e^x + 5e^{3x}$$

is also a solution of differential equation (A) on $a \le x \le b$?

(c) Each of the functions

$$3e^x, \qquad -4e^x, \qquad 5e^x, \quad \text{and} \quad 6e^x$$

is also a solution of differential equation (A) on $a \le x \le b$. Why?

5. Again consider the differential equation (A) of Exercise 4.

(a) Use the definition of linear dependence to show that the four functions of part (c) of Exercise 4 are linearly dependent on $a \le x \le b$.

(b) Use Theorem 4.4 to show that each pair of the four solutions of differential equation (A) listed in part (c) of Exercise 4 are linearly dependent on $a \le x \le b$.

6. Again consider the differential equation (A) of Exercise 4.

(a) Use the definition of linear independence to show that the two functions e^x and e^{3x} are linearly independent on $a \le x \le b$.

(b) Use Theorem 4.4 to show that the two solutions e^x and e^{3x} of differential equation (A) are linearly independent on $a \le x \le b$.

7. Consider the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

(a) Show that e^{2x} and e^{3x} are linearly independent solutions of this equation on the interval $-\infty < x < \infty$.

(b) Write the general solution of the given equation.

(c) Find the solution that satisfies the conditions $y(0) = 2$, $y'(0) = 3$. Explain why this solution is unique. Over what interval is it defined?

8. Consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0.$$

(a) Show that e^x and xe^x are linearly independent solutions of this equation on the interval $-\infty < x < \infty$.

(b) Write the general solution of the given equation.

(c) Find the solution that satisfies the condition $y(0) = 1$, $y'(0) = 4$. Explain why this solution is unique. Over what interval is it defined?

4.5 THE CAUCHY–EULER EQUATION

A. The Equation and the Method of Solution

In the preceding sections we have seen how to obtain the general solution of the nth-order linear differential equation with *constant* coefficients. We have seen that in such cases the form of the complementary function may be readily determined. The general nth-order linear equation with *variable* coefficients is quite a different matter, however, and only in certain special cases can the complementary function be obtained explicity in closed form. One special case of considerable practical importance for which it is fortunate that this can be done is the so-called *Cauchy-Euler equation* (or *equi-dimensional* equation). This is an equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x), \tag{4.84}$$

where $a_0, a_1, \ldots, a_{n-1}$, a_n are constants. Note the characteristic feature of this equation: each term in the left member is a constant multiple of an expression of the form

$$x^k \frac{d^k y}{dx^k}.$$

How should one proceed to solve such an equation? About the only hopeful thought that comes to mind at this stage of our study is to attempt a transformation. But what transformation should we attempt and where will it lead us? While it is certainly worthwhile to stop for a moment and consider what sort of transformation we might use in solving a "new" type of equation when we first encounter it, it is certainly not worthwhile to spend a great deal of time looking for clever devices which mathematicians have known about for many years. The facts are stated in the following theorem.

THEOREM 4.14

The transformation $x = e^t$ reduces the equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x) \tag{4.84}$$

to a linear differential equation with constant coefficients.

This is what we need! We shall prove this theorem for the case of the *second*-order Cauchy–Euler differential equation

$$a_0 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = F(x). \tag{4.85}$$

The proof in the general nth-order case proceeds in a similar fashion. Letting $x = e^t$, assuming $x > 0$, we have $t = \ln x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

and

$$\frac{d^2 y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x}\left(\frac{d^2 y}{dt^2}\frac{dt}{dx}\right) - \frac{1}{x^2}\frac{dy}{dt}$$
$$\frac{1}{x}\left(\frac{d^2 y}{dt^2}\frac{1}{x}\right) - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right).$$

Thus

$$x\frac{dy}{dx} = \frac{dy}{dt} \quad \text{and} \quad x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}.$$

Substituting into Equation (4.85) we obtain

$$a_0\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right) + a_1 \frac{dy}{dt} + a_2 y = F(e^t)$$

or

$$A_0 \frac{d^2 y}{dt^2} + A_1 \frac{dy}{dt} + A_2 y = G(t), \tag{4.86}$$

where

$$A_0 = a_0, \qquad A_1 = a_1 - a_0, \qquad A_2 = a_2, \qquad G(t) = F(e^t).$$

This is a second-order linear differential equation with *constant* coefficients, which was what we wished to show.

Remarks. 1. Note that the leading coefficient $a_0 x^n$ in Equation (4.84) is zero for $x = 0$. Thus the basic interval $a \le x \le b$, referred to in the general theorems of Section 4.1, does *not* include $x = 0$.

2. Observe that in the above proof we assumed that $x > 0$. If $x < 0$, the substitution $x = -e^t$ is actually the correct one. Unless the contrary is explicitly stated, we shall assume $x > 0$ when finding the general solution of a Cauchy–Euler differential equation.

B. Examples

▶ Example 4.43

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3. \tag{4.87}$$

Let $x = e^t$. Then, assuming $x > 0$, we have $t = \ln x$, and

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{d^2y}{dt^2}\frac{dt}{dx}\right) - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Thus Equation (4.87) becomes

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t}$$

or

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}. \tag{4.88}$$

The complementary function of this equation is $y_c = c_1 e^t + c_2 e^{2t}$. We find a particular integral by the method of undetermined coefficients. We assume $y_p = Ae^{3t}$. Then $y_p' = 3Ae^{3t}$, $y_p'' = 9Ae^{3t}$, and substituting into Equation (4.88) we obtain

$$2Ae^{3t} = e^{3t}.$$

Thus $A = \frac{1}{2}$ and we have $y_p = \frac{1}{2}e^{3t}$. The general solution of Equation (4.88) is then

$$y = c_1 e^t + c_2 e^{2t} + \tfrac{1}{2}e^{3t}.$$

But we are not yet finished! We must return to the original independent variable x. Since $e^t = x$, we find

$$y = c_1 x + c_2 x^2 + \tfrac{1}{2}x^3.$$

This is the general solution of Equation (4.87).

Remarks. 1. Note carefully that under the transformation $x = e^t$ the right member of (4.87), x^3, transforms into e^{3t}. The student should be careful to transform *both* sides of the equation if he intends to obtain a particular integral of the given equation by finding a particular integral of the transformed equation, as we have done here.

2. We hasten to point out that the following alternative procedure may be used. After finding the complementary function of the transformed equation one can immediately write the complementary function of the original given equation and then proceed to obtain a particular integral of the original equation by variation of parameters. In Example 4.43, upon finding the complementary function $c_1e^t + c_2e^{2t}$ of Equation (4.88), one can immediately write the complementary function $c_1x + c_2x^2$ of Equation (4.87), then assume the particular integral $y_p(x) = v_1(x)x + v_2(x)x^2$, and from here proceed by the method of variation of parameters. However, when the nonhomogeneous function F transforms into a linear combination of UC functions, as it does in this example, the procedure illustrated is generally simpler.

▶ Example 4.44

$$x^3\frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} - 8y = 4\ln x. \tag{4.89}$$

Assuming $x > 0$, we let $x = e^t$. Then $t = \ln x$, and

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Now we must consider $\dfrac{d^3y}{dx^3}$.

$$\begin{aligned}\frac{d^3y}{dx^3} &= \frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\\ &= \frac{1}{x^2}\left(\frac{d^3y}{dt^3}\frac{dt}{dx} - \frac{d^2y}{dt^2}\frac{dt}{dx}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\\ &= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2}\right) - \frac{2}{x^3}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\\ &= \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right).\end{aligned}$$

Thus, substituting into Equation (4.89), we obtain

$$\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) - 4\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + 8\left(\frac{dy}{dt}\right) - 8y = 4t$$

or

$$\frac{d^3y}{dt^3} - 7\frac{d^2y}{dt^2} + 14\frac{dy}{dt} - 8y = 4t. \tag{4.90}$$

The complementary function of the transformed equation (4.90) is

$$y_c = c_1e^t + c_2e^{2t} + c_3e^{4t}.$$

We procced to obtain a particular integral of Equation (4.90) by the method of undetermined coefficients. We assume $y_p = At + B$. Then $y'_p = A$, $y''_p = y'''_p = 0$. Substituting into Equation (4.90), we find

$$14A - 8At - 8B = 4t.$$

Thus

$$-8A = 4, \qquad 14A - 8B = 0,$$

and so $A = -\frac{1}{2}$, $B = -\frac{7}{8}$. Thus the general solution of Equation (4.90) is

$$y = c_1 e^t + c_2 e^{2t} + c_3 e^{4t} - \tfrac{1}{2}t - \tfrac{7}{8},$$

and so the general solution of Equation (4.89) is

$$y = c_1 x + c_2 x^2 + c_3 x^4 - \tfrac{1}{2}\ln x - \tfrac{7}{8}.$$

Remarks. In solving the Cauchy–Euler equations of the preceding examples, we observe that the transformation $x = e^t$ reduces

$$x\frac{dy}{dx} \quad \text{to} \quad \frac{dy}{dt}, \qquad x^2\frac{d^2y}{dx^2} \quad \text{to} \quad \frac{d^2y}{dt^2} - \frac{dy}{dt},$$

and

$$x^3\frac{d^3y}{dx^3} \quad \text{to} \quad \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}.$$

We now show (without proof) how to find the expression into which the general term

$$x^n\frac{d^ny}{dx^n},$$

where n is an *arbitrary* positive integer, reduces under the transformation $x = e^t$. We present this as the following formal four-step procedure.

1. For the given positive integer n, determine

$$r(r-1)(r-2)\cdots[r-(n-1)].$$

2. Expand the preceding as a polynomial of degree n in r.
3. Replace r^k by $\dfrac{d^ky}{dt^k}$, for each $k = 1, 2, 3, \ldots, n$.
4. Equate $x^n\dfrac{d^ny}{dx^n}$ to the result in Step 3.

For example, when $n = 3$, we have the following illustration.

1. Since $n = 3$, $n - 1 = 2$ and we determine $r(r-1)(r-2)$.
2. Expanding the preceding, we obtain $r^3 - 3r^2 + 2r$.
3. Replacing r^3 by $\dfrac{d^3y}{dt^3}$, r^2 by $\dfrac{d^2y}{dt^2}$, and r by $\dfrac{dy}{dt}$, we have

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}.$$

THEOREM 4.17

Two solutions f_1 and f_2 of the second-order homogeneous linear differential equation (4.91) are linear independent on $a \leq x \leq b$ if and only if the value of the Wronskian of f_1 and f_2 is different from zero for some x on the interval $a \leq x \leq b$.

Method of Proof. We prove this theorem by proving the following equivalent theorem.

THEOREM 4.18

Two solutions f_1 and f_2 of the second-order homogeneous linear differential equation (4.91) are linearly dependent on $a \leq x \leq b$ if and only if the value of the Wronskian of f_1 and f_2 is zero for all x on $a \leq x \leq b$:

$$\begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix} = 0 \quad \textit{for all } x \textit{ on } a \leq x \leq b.$$

Proof. Part 1. We must show that if the value of the Wronskian of f_1 and f_2 is zero for all x on $a \leq x \leq b$, then f_1 and f_2 are linearly dependent on $a \leq x \leq b$. We thus assume that

$$\begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix} = 0$$

for all x such that $a \leq x \leq b$. Then at any particular x_0 such that $a \leq x_0 \leq b$, we have

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f'_1(x_0) & f'_2(x_0) \end{vmatrix} = 0.$$

Thus, by Theorem B, there exist constants c_1 and c_2, not both zero, such that

$$\begin{aligned} c_1 f_1(x_0) + c_2 f_2(x_0) &= 0, \\ c_1 f'_1(x_0) + c_2 f'_2(x_0) &= 0. \end{aligned} \tag{4.101}$$

Now consider the function f defined by

$$f(x) = c_1 f_1(x) + c_2 f_2(x), \qquad a \leq x \leq b.$$

By Theorem 4.15, since f_1 and f_2 are solutions of differential equation (4.91), this function f is also a solution of Equation (4.91). From (4.101), we have

$$f(x_0) = 0 \quad \text{and} \quad f'(x_0) = 0.$$

Thus by Theorem A, Conclusion 2, we know that

$$f(x) = 0 \quad \text{for all } x \text{ on } a \leq x \leq b.$$

That is,

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x on $a \leq x \leq b$, where c_1 and c_2 are not both zero. Therefore the solutions f_1 and f_2 are linearly dependent on $a \leq x \leq b$.

Part 2. We must now show that if f_1 and f_2 are linearly dependent on $a \leq x \leq b$, then their Wronskian has the value zero for all x on this interval. We thus assume that f_1 and f_2 are linearly dependent on $a \leq x \leq b$. Then there exist constants c_1 and c_2, not both zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0 \tag{4.102}$$

for all x on $a \leq x \leq b$. From (4.102), we also have

$$c_1 f'_1(x) + c_2 f'_2(x) = 0 \tag{4.103}$$

for all x on $a \leq x \leq b$. Now let $x = x_0$ be an arbitrary point of the interval $a \leq x \leq b$. Then (4.102) and (4.103) hold at $x = x_0$. That is,

$$\begin{aligned} c_1 f_1(x_0) + c_2 f_2(x_0) &= 0, \\ c_1 f'_1(x_0) + c_2 f'_2(x_0) &= 0, \end{aligned}$$

where c_1 and c_2 are not both zero. Thus, by Theorem B, we have

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f'_1(x_0) & f'_2(x_0) \end{vmatrix} = 0.$$

But this determinant is the value of the Wronskian of f_1 and f_2 at $x = x_0$, and x_0 is an *arbitrary* point of $a \le x \le b$. Thus we have

$$\begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix} = 0$$

for *all* x on $a \le x \le b$. *Q.E.D.*

THEOREM 4.19

The value of the Wronskian of two solutions f_1 and f_2 of differential equation (4.91) either is zero for all x on $a \le x \le b$ or is zero for no x on $a \le x \le b$.

Proof. If f_1 and f_2 are linearly dependent on $a \le x \le b$, then by Theorem 4.18, the value of the Wronskian of f_1 and f_2 is zero for all x on $a \le x \le b$.

Now let f_1 and f_2 be linearly independent on $a \le x \le b$; and let W denote the Wronskian of f_1 and f_2, so that

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix}.$$

Differentiating this, we obtain

$$W'(x) = \begin{vmatrix} f'_1(x) & f'_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ f''_1(x) & f''_2(x) \end{vmatrix},$$

and this reduces at once to

$$W'(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f''_1(x) & f''_2(x) \end{vmatrix}. \tag{4.104}$$

Since f_1 and f_2 are solutions of differential equation (4.91), we have, respectively,

$$a_0(x)f''_1(x) + a_1(x)f'_1(x) + a_2(x)f_1(x) = 0,$$

$$a_0(x)f''_2(x) + a_1(x)f'_2(x) + a_2(x)f_2(x) = 0,$$

and hence

$$f''_1(x) = -\frac{a_1(x)}{a_0(x)} f'_1(x) - \frac{a_2(x)}{a_0(x)} f_1(x),$$

$$f''_2(x) = -\frac{a_1(x)}{a_0(x)} f'_2(x) - \frac{a_2(x)}{a_0(x)} f_2(x)$$

on $a \le x \le b$. Substituting these expressions into (4.104), we obtain

$$W'(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ -\dfrac{a_1(x)}{a_0(x)} f'_1(x) - \dfrac{a_2(x)}{a_0(x)} f_1(x) & -\dfrac{a_1(x)}{a_0(x)} f'_2(x) - \dfrac{a_2(x)}{a_0(x)} f_2(x) \end{vmatrix}.$$

This reduces at once to

$$W'(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ -\dfrac{a_1(x)}{a_0(x)} f'_1(x) & -\dfrac{a_1(x)}{a_0(x)} f'_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ -\dfrac{a_2(x)}{a_0(x)} f_1(x) & -\dfrac{a_2(x)}{a_0(x)} f_2(x) \end{vmatrix},$$

and since the last determinant has two proportional rows, this in turn reduces to

$$W'(x) = -\frac{a_1(x)}{a_0(x)}\begin{vmatrix} f_1(x) & f_2(x) \\ f'_1(x) & f'_2(x) \end{vmatrix},$$

which is simply

$$W'(x) = -\frac{a_1(x)}{a_0(x)}W(x).$$

Thus the Wronskian W satisfies the first-order homogeneous linear differential equation

$$\frac{dW}{dx} + \frac{a_1(x)}{a_0(x)}W = 0.$$

Integrating this from x_0 to x, where x_0 is an arbitrary point of $a \le x \le b$, we obtain

$$W(x) = c\exp\left[-\int_{x_0}^{x}\frac{a_1(t)}{a_0(t)}\,dt\right].$$

Letting $x = x_0$, we find that $c = W(x_0)$. Hence we obtain the identity

$$W(x) = W(x_0)\exp\left[-\int_{x_0}^{x}\frac{a_1(t)}{a_0(t)}\,dt\right], \tag{4.105}$$

valid for all x on $a \le x \le b$, where x_0 is an arbitrary point of this interval.

Now assume that $W(x_0) = 0$. Then by identity (4.105), we have $W(x) = 0$ for all x on $a \le x \le b$. Thus by Theorem 4.18, the solutions f_1 and f_2 must be linearly dependent on $a \le x \le b$. This is a contradiction, since f_1 and f_2 are linearly independent. Therefore the assumption that $W(x_0) = 0$ is false, and so $W(x_0) \ne 0$. But x_0 is an *arbitrary* point of $a \le x \le b$. Thus $W(x)$ is zero for no x on $a \le x \le b$.

Q.E.D.

THEOREM 4.20

Hypothesis. *Let f_1 and f_2 be any two linearly independent solutions of differential equation (4.91) on $a \le x \le b$.*

Conclusion. *Then every solution f of differential equation (4.91) can be expressed as a suitable linear combination*

$$c_1 f_1 + c_2 f_2$$

of these two linear independent solutions.

Proof. Let x_0 be an arbitrary point of the interval $a \le x \le b$, and consider the following system of two linear algebraic equations in the two unknowns k_1 and k_2:

$$\begin{aligned} k_1 f_1(x_0) + k_2 f_2(x_0) &= f(x_0), \\ k_1 f'_1(x_0) + k_2 f'_2(x_0) &= f'(x_0). \end{aligned} \tag{4.106}$$

Since f_1 and f_2 are linearly independent on $a \le x \le b$, we know by Theorem 4.17 that the value of the Wronskian of f_1 and f_2 is different from zero at some point of this interval. Then by Theorem 4.19 the value of the Wronskian is zero for no x on $a \le x \le b$ and hence its value at x_0 is not zero. That is,

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f'_1(x_0) & f'_2(x_0) \end{vmatrix} \neq 0.$$

Thus by Theorem C, the algebraic system (4.106) has a unique solution $k_1 = c_1$ and $k_2 = c_2$. Thus for $k_1 = c_1$ and $k_2 = c_2$, each left member of system (4.106) is the same number as the corresponding right member of (4.106). That is, the number $c_1 f_1(x_0) + c_2 f_2(x_0)$ is equal to the number $f(x_0)$, and the number $c_1 f'_1(x_0) + c_2 f'_2(x_0)$ is equal to the number $f'(x_0)$. But the numbers $c_1 f_1(x_0) + c_2 f_2(x_0)$ and $c_1 f'_1(x_0) + c_2 f'_2(x_0)$ are the values of the solution $c_1 f_1 + c_2 f_2$ and its first derivative, respectively, at x_0; and the numbers $f(x_0)$ and $f'(x_0)$ are the values of the solution f and its first derivative, respectively, at x_0. Thus the two solutions $c_1 f_1 + c_2 f_2$ and f have equal values and their first derivative also have equal values at x_0. Hence by Theorem A, Conclusion 1, we know that these two solutions are identical throughout the interval $a \le x \le b$. That is,

$$f(x) = c_1 f_1(x) + c_2 f_2(x)$$

for all x on $a \le x \le b$, and so f is expressed as a linear combination of f_1 and f_2.

Q.E.D.

Exercises

1. Consider the second-order homogenous linear differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

   (a) Find the two linearly independent solutions f_1 and f_2 of this equation which are such that

$$f_1(0) = 1 \quad \text{and} \quad f'_1(0) = 0$$

   and

$$f_2(0) = 0 \quad \text{and} \quad f'_2(0) = 1.$$

   (b) Express the solution

$$3e^x + 2e^{2x}$$

   as a linear combination of the two linearly independent solutions f_1 and f_2 defined in part (a).

2. Consider the second-order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, \tag{A}$$

   where a_0, a_1, and a_2 are continuous on a real interval $a \le x \le b$, and $a_0(x) \neq 0$ for all x on this interval. Let f_1 and f_2 be two distinct solutions of differential equation (A) on $a \le x \le b$, and suppose $f_2(x) \neq 0$ for all x on this interval. Let $W[f_1(x), f_2(x)]$ be the value of the Wronskian of f_1 and f_2 at x.